

PENTAVALENT SYMMETRIC GRAPHS OF ORDER FOUR TIMES AN ODD SQUARE-FREE INTEGER

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ABSTRACT. A graph is said to be symmetric if its automorphism group is transitive on its arcs. Guo et al. (Electronic J. Combin. 18, #P233, 2011) and Pan et al. (Electronic J. Combin. 20, #P36, 2013) determined all pentavalent symmetric graphs of order $4pq$. In this paper, we shall generalize this result by determining all connected pentavalent symmetric graphs of order four times an odd square-free integer. It is shown in this paper that, for each of such graphs Γ , either the full automorphism group $\text{Aut}\Gamma$ is isomorphic to $\text{PSL}(2, p)$, $\text{PGL}(2, p)$, $\text{PSL}(2, p) \times \mathbb{Z}_2$ or $\text{PGL}(2, p) \times \mathbb{Z}_2$, or Γ is isomorphic to one of 8 graphs.

Keywords. Arc-transitive graph; Normal quotient; Automorphism group.

1. INTRODUCTION

All graphs in this paper are assumed to be finite, simple, connected and undirected. Let Γ be a graph and denote $V\Gamma$ and $A\Gamma$ the vertex set and arc set of Γ , respectively. Let G be a subgroup of the full automorphism group $\text{Aut}\Gamma$ of Γ . Then Γ is called G -vertex-transitive and G -arc-transitive if G is transitive on $V\Gamma$ and $A\Gamma$, respectively. An arc-transitive graph is also called *symmetric*. It is well known that Γ is G -arc-transitive if and only if G is transitive on $V\Gamma$ and the stabilizer $G_\alpha := \{g \in G \mid \alpha^g = \alpha\}$ for some $\alpha \in V\Gamma$ is transitive on the neighbor set $\Gamma(\alpha)$ of α in Γ .

The cubic and tetravalent graphs have been studied extensively in the literature. In recent years, attention has moved on to pentavalent symmetric graphs and a series of results have been obtained. For example, all the possibilities of vertex stabilizers of pentavalent symmetric graphs are determined in [7, 18]. Also, for distinct primes p , q and r , the classifications of pentavalent symmetric graphs of order $2pq$ and $2pqr$ are presented in [9, 17], respectively. A classification of 1-regular pentavalent graph (that is, the full automorphism group acts regularly on its arc set) of square-free order is presented in [13]. Recently, pentavalent symmetric graphs of square-free order have been completely classified in [11]. Furthermore, some classifications of pentavalent symmetric graphs of cube-free order also have been obtained in recent years. For example, the classifications of pentavalent symmetric graphs of order $12p$, $4pq$ and $2p^2$ are presented in [5, 8, 14]. The main purpose of this paper is to extend the results in [8, 14] to four times an odd square-free integer case.

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The main result of this paper is the following theorem.

Theorem 1.1. *Let n be an odd square-free integer and let Γ be a pentavalent symmetric graph of order $4n$. If n has at least three prime factors, then one of the following statements holds.*

- (1) $\text{Aut}\Gamma \cong \text{PSL}(2, p), \text{PGL}(2, p), \text{PSL}(2, p) \times \mathbb{Z}_2$ or $\text{PGL}(2, p) \times \mathbb{Z}_2$, where $p \geq 29$ is a prime.
- (2) The triple $(\Gamma, n, \text{Aut}\Gamma)$ lies in the following Table 1.

Row	Γ	n	$\text{Aut}\Gamma$	$(\text{Aut}\Gamma)_\alpha$	Transitivity	Bipartite?
1	\mathcal{C}_{17556}^1	$3 \cdot 7 \cdot 11 \cdot 19$	J_1	D_{10}	1 – transitive	No
2	\mathcal{C}_{17556}^2	$3 \cdot 7 \cdot 11 \cdot 19$	J_1	D_{10}	1 – transitive	No
3	\mathcal{C}_{17556}^3	$3 \cdot 7 \cdot 11 \cdot 19$	J_1	D_{10}	1 – transitive	No
4	\mathcal{C}_{17556}^4	$3 \cdot 7 \cdot 11 \cdot 19$	J_1	D_{10}	1 – transitive	No
5	\mathcal{C}_{17556}^5	$3 \cdot 7 \cdot 11 \cdot 19$	J_1	D_{10}	1 – transitive	No
6	\mathcal{C}_{5852}	$7 \cdot 11 \cdot 19$	$J_1 \times \mathbb{Z}_2$	A_5	2 – transitive	Yes
7	\mathcal{C}_{780}^1	$3 \cdot 5 \cdot 13$	$\text{PSL}(2, 25) \times \mathbb{Z}_2$	F_{20}	2 – transitive	No
8	\mathcal{C}_{780}^2	$3 \cdot 5 \cdot 13$	$\text{PSL}(2, 25) \times \mathbb{Z}_2$	F_{20}	2 – transitive	No

TABLE 1.

Remark 1.1.

- (a) The graphs in Table 1 are introduced in Example 3.1.
- (b) It seems not feasible to determine all the possible values of p in part (1) for general odd square-free integer n . However, if the number of the prime divisors of n is fixed, then it is not difficult to determine the possible values of p and hence all corresponding graphs Γ .

2. PRELIMINARIES

We now give some necessary preliminary results. The first one is a property of the Fitting subgroup, see [16, P. 30, Corollary].

Lemma 2.1. *Let F be the Fitting subgroup of a group G . If G is soluble, then $F \neq 1$ and the centralizer $C_G(F) \leq F$.*

The maximal subgroups of $\text{PSL}(2, q)$ are known, see [4, Section 239].

Lemma 2.2. *Let $T = \text{PSL}(2, q)$, where $q = p^n \geq 5$ with p a prime. Then a maximal subgroup of T is isomorphic to one of the following groups, where $d = (2, q - 1)$.*

- (1) $D_{2(q-1)/d}$, where $q \neq 5, 7, 9, 11$;
- (2) $D_{2(q+1)/d}$, where $q \neq 7, 9$;
- (3) $\mathbb{Z}_p^n : \mathbb{Z}_{(q-1)/d}$;
- (4) A_4 , where $q = p = 5$ or $q = p \equiv 3, 13, 27, 37 \pmod{40}$;
- (5) S_4 , where $q = p \equiv \pm 1 \pmod{8}$;
- (6) A_5 , where $q = p \equiv \pm 1 \pmod{5}$, or $q = p^2 \equiv -1 \pmod{5}$ with p an odd prime;

- (7) $\text{PSL}(2, p^m)$ with n/m an odd integer;
- (8) $\text{PGL}(2, p^{n/2})$ with n an even integer.

By [2, Theorem 2], we may easily derive the maximal subgroups of $\text{PGL}(2, p)$.

Lemma 2.3. *Let $T = \text{PGL}(2, p)$ with $p \geq 5$ a prime. Then a maximal subgroup of T is isomorphic to one of the following groups:*

- (1) $\mathbb{Z}_p : \mathbb{Z}_{p-1}$;
- (2) $D_{2(p+1)}$;
- (3) $D_{2(p-1)}$, where $p \geq 7$;
- (4) S_4 , where $p \equiv \pm 3 \pmod{8}$;
- (5) $\text{PSL}(2, p)$.

For a graph Γ and a positive integer s , an s -arc of Γ is a sequence $\alpha_0, \alpha_1, \dots, \alpha_s$ of vertices such that α_{i-1}, α_i are adjacent for $1 \leq i \leq s$ and $\alpha_{i-1} \neq \alpha_{i+1}$ for $1 \leq i \leq s-1$. In particular, a 1-arc is just an arc. Then Γ is called (G, s) -arc-transitive with $G \leq \text{Aut}\Gamma$ if G is transitive on the set of s -arcs of Γ . A (G, s) -arc-transitive graph is called (G, s) -transitive if it is not $(G, s+1)$ -arc-transitive. In particular, a graph Γ is simply called s -transitive if it is $(\text{Aut}\Gamma, s)$ -transitive.

Let F_{20} denote the Frobenius group of order 20. The following lemma determines the stabilizers of pentavalent symmetric graphs, refer to [7, 18].

Lemma 2.4. *Let Γ be a pentavalent (G, s) -transitive graph, where $G \leq \text{Aut}\Gamma$ and $s \geq 1$. Let $\alpha \in V\Gamma$. Then one of the following holds.*

- (a) *If G_α is soluble, then $s \leq 3$ and $|G_\alpha| \mid 80$. Further, the couple (s, G_α) lies in the following table.*

s	1	2	3
G_α	$\mathbb{Z}_5, D_{10}, D_{20}$	$F_{20}, F_{20} \times \mathbb{Z}_2$	$F_{20} \times \mathbb{Z}_4$

- (b) *If G_α is insoluble, then $2 \leq s \leq 5$, and $|G_\alpha| \mid 2^9 \cdot 3^2 \cdot 5$. Further, the couple (s, G_α) lies in the following table.*

s	2	3	4	5
G_α	A_5, S_5	$A_4 \times A_5, (A_4 \times A_5) : \mathbb{Z}_2,$ $S_4 \times S_5$	$\text{ASL}(2, 4), \text{AGL}(2, 4),$ $\text{A}\Sigma\text{L}(2, 4), \text{A}\Gamma\text{L}(2, 4)$	$\mathbb{Z}_2^6 : \Gamma\text{L}(2, 4)$
$ G_\alpha $	60, 120	720, 1440, 2880	960, 1920, 2880, 5760	23040

From [6, pp. 134-136], we can obtain the following lemma by checking the orders of nonabelian simple groups.

Lemma 2.5. *Let n be an odd square-free integer such that n has at least three prime factors. Let T be a nonabelian simple group of order $2^i \cdot 3^j \cdot 5 \cdot n$, where $1 \leq i \leq 11$ and $0 \leq j \leq 2$. Let p be the largest prime factor of n . Then T is listed Table 2.*

Proof. If T is a sporadic simple group, by [6, P. 135-136], $T = M_{22}, M_{23}, M_{24}, J_1$ or J_2 . If $T = A_n$ is an alternating group, since 3^4 does not divide $|T|$, we have $n \leq 8$, it then easily exclude that $T = A_5, A_6, A_7$ or A_8 . Hence no T exists for this case.

T	$ T $	n	T	$ T $	n
M_{22}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	$3 \cdot 7 \cdot 11$	$\text{PSp}(4, 4)$	$2^8 \cdot 3^2 \cdot 5^2 \cdot 17$	$3 \cdot 5 \cdot 17$
M_{23}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	$7 \cdot 11 \cdot 23$	$\text{PSL}(2, 25)$	$2^3 \cdot 3 \cdot 5^2 \cdot 13$	$3 \cdot 5 \cdot 13$
J_1	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$	$7 \cdot 11 \cdot 19$	$\text{PSL}(2, 2^8)$	$2^8 \cdot 3 \cdot 5 \cdot 17 \cdot 257$	$3 \cdot 17 \cdot 257$
J_2	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$	$3 \cdot 5 \cdot 7$	$\text{PSL}(5, 2)$	$2^{10} \cdot 3^2 \cdot 5 \cdot 7 \cdot 31$	$3 \cdot 7 \cdot 31$
$\text{Sz}(32)$	$2^{10} \cdot 5^2 \cdot 31 \cdot 41$	$5 \cdot 31 \cdot 41$	$\text{PSL}(2, 2^6)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$	$3 \cdot 7 \cdot 13$
$\text{PSU}(3, 4)$	$2^6 \cdot 3 \cdot 5^2 \cdot 13$	$3 \cdot 5 \cdot 13$			
M_{23}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	$3 \cdot 7 \cdot 11 \cdot 23$	J_1	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$	$3 \cdot 7 \cdot 11 \cdot 19$
M_{24}	$2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	$3 \cdot 7 \cdot 11 \cdot 23$			
$\text{PSL}(2, p)$	$\frac{p(p+1)(p-1)}{2}$	$(p \geq 29)$			

TABLE 2.

Suppose now $T = X(q)$ is a simple group of Lie type, where X is one type of Lie groups, and $q = r^d$ is a prime power. If $r \geq 5$, as $|T|$ has at most three 3-factors, two 5-factors and one p -factor, it easily follows from [6, P. 135] that the only possibility is $T = \text{PSL}(2, p)$ with $p \geq 29$ (note that $\text{PSL}(2, p)$ with $5 \leq p \leq 23$ does not satisfy the condition of the lemma) or $\text{PSL}(2, 25)$, where p is the largest prime factor of n . If $r \leq 3$, as 2^{12} and 3^4 do not divide $|T|$, then we have $T = \text{Sz}(32)$, $\text{PSU}(3, 4)$, $\text{PSp}(4, 4)$, $\text{PSL}(2, 2^6)$, $\text{PSL}(2, 2^8)$ or $\text{PSL}(5, 2)$. \square

A typical method for studying vertex-transitive graphs is taking normal quotients. Let Γ be a G -vertex-transitive graph, where $G \leq \text{Aut}\Gamma$. Suppose that G has a normal subgroup N which is intransitive on $V\Gamma$. Let $V\Gamma_N$ be the set of N -orbits on $V\Gamma$. The *normal quotient graph* Γ_N of Γ induced by N is defined as the graph with vertex set $V\Gamma_N$, and B is adjacent to C in Γ_N if and only if there exist vertices $\beta \in B$ and $\gamma \in C$ such that β is adjacent to γ in Γ . In particular, if $\text{val}(\Gamma) = \text{val}(\Gamma_N)$, then Γ is called a *normal cover* of Γ_N .

A graph Γ is called *G -locally primitive* if, for each $\alpha \in V\Gamma$, the stabilizer G_α acts primitively on $\Gamma(\alpha)$. Obviously, a pentavalent symmetric graph is locally primitive. The following theorem gives a basic method for studying vertex-transitive locally primitive graphs, see [15, Theorem 4.1] and [12, Lemma 2.5].

Theorem 2.6. *Let Γ be a G -vertex-transitive locally primitive graph, where $G \leq \text{Aut}\Gamma$, and let $N \triangleleft G$ have at least three orbits on $V\Gamma$. Then the following statements hold.*

- (i) N is semi-regular on $V\Gamma$, $G/N \leq \text{Aut}\Gamma_N$, and Γ is a normal cover of Γ_N ;
- (ii) $G_\alpha \cong (G/N)_\gamma$, where $\alpha \in V\Gamma$ and $\gamma \in V\Gamma_N$;
- (iii) Γ is (G, s) -transitive if and only if Γ_N is $(G/N, s)$ -transitive, where $1 \leq s \leq 5$ or $s = 7$.

For reduction, we need some information of pentavalent symmetric graphs of order $4pq$, stated in the following lemma, see [8, Theorem 4.1] and [14, Theorem 3.1].

Lemma 2.7. *Let Γ be a pentavalent symmetric graph of order $4pq$, where $q > p \geq 3$ are primes. Then the couple $(\text{Aut}\Gamma, (\text{Aut}\Gamma)_\alpha)$ lies in the following Table 3, where $\alpha \in V\Gamma$.*

Γ	(p, q)	$\text{Aut}\Gamma$	$(\text{Aut}\Gamma)_\alpha$
\mathcal{C}_{60}	$(3, 5)$	$A_5 \times D_{10}$	D_{10}
\mathcal{C}_{132}^1	$(3, 11)$	$\text{PSL}(2, 11) \times \mathbb{Z}_2$	D_{10}
$\mathcal{C}_{132}^i, 2 \leq i \leq 4$	$(3, 11)$	$\text{PGL}(2, 11)$	D_{10}
\mathcal{C}_{132}^5	$(3, 11)$	$\text{PGL}(2, 11) \times \mathbb{Z}_2$	D_{20}
$\mathcal{C}_{574}^{(2)}$	$(7, 41)$	$\text{PSL}(2, 41) \times \mathbb{Z}_2$	A_5
\mathcal{C}_{4108}	$(13, 79)$	$\text{PSL}(2, 79)$	A_5

TABLE 3.

Remark 2.8.

- (a) Suppose that Γ is one of the graphs in Lemma 2.7 and M is an arc-transitive subgroup of $\text{Aut}\Gamma$. Then M is insoluble (for convenience, we prove this conclusion in Lemma 4.4 and we remark that Lemma 4.4 is independent where it is used).
- (b) By Magma [1], the graphs $\mathcal{C}_{66}^{(2)}$ and \mathcal{C}_{132}^5 in [8, Theorem 4.1] are isomorphic, $\text{Aut}(\mathcal{C}_{132}^5) \cong \text{PGL}(2, 11) \times \mathbb{Z}_2$.

The final lemma of this section gives some information about the pentavalent symmetric graphs of square-free order, refer to [17, Theorem 1.1] and [11, Theorem 1.1].

Lemma 2.8. *Let Γ be a pentavalent symmetric graph of order $2n$, where n is an odd square-free integer and has at least three prime factors. Then one of the following statements holds.*

- (1) $\text{Aut}\Gamma$ is soluble and $\text{Aut}\Gamma \cong D_{2n} : \mathbb{Z}_5$.
- (2) $\text{Aut}\Gamma = \text{PSL}(2, p)$ or $\text{PGL}(2, p)$, where $p \geq 5$ is a prime.
- (3) The triple $(\Gamma, 2n, \text{Aut}\Gamma)$ lies in the following Table 4.

Γ	$2n$	$\text{Aut}\Gamma$	$(\text{Aut}\Gamma)_\alpha$
\mathcal{C}_{390}	390	$\text{PSL}(2, 25)$	F_{20}
\mathcal{C}_{2926}	2926	J_1	A_5

TABLE 4.

3. SOME EXAMPLES

In this section, we give some examples of pentavalent symmetric graphs of order $4n$ with n an odd square-free integer.

For a given small permutation group X , we may determine all graphs which admit X as an arc-transitive automorphism group by using Magma [1]. It is then easy to have the following result.

Example 3.1. (1) There is a unique pentavalent symmetric graph of order 5852 which admits $J_1 \times \mathbb{Z}_2$ as an arc-transitive automorphism group; and its full automorphism group is $J_1 \times \mathbb{Z}_2$. This graph is denoted by \mathcal{C}_{5832} which satisfies the conditions in Row 6 of Table 1.

(2) There are five pentavalent symmetric graphs of order 17556 admitting J_1 as an arc-transitive automorphism group; and their full automorphism group are all isomorphic to J_1 . These five graphs are denoted by \mathcal{C}_{17556}^i which satisfy the conditions in Row 1 to Row 5 of Table 1, where $1 \leq i \leq 5$.

(3) There are two pentavalent symmetric graphs of order 780 which admits $\text{PSL}(2, 25) \times \mathbb{Z}_2$ as an arc-transitive automorphism group; and their full automorphism group are all isomorphic to $\text{PSL}(2, 25) \times \mathbb{Z}_2$. These two graphs are denoted by \mathcal{C}_{780}^j which satisfy the conditions in Row 7 to Row 8 of Table 1, where $1 \leq j \leq 2$.

4. PROOF OF THEOREM 1.1

Let n be an odd square-free integer and n has at least three prime factors. Let Γ be a pentavalent symmetric graph of order $4n$. Set $A = \text{Aut}\Gamma$. By Lemma 2.4, $|A_\alpha| \mid 2^9 \cdot 3^2 \cdot 5$, and hence $|A| \mid 2^{11} \cdot 3^2 \cdot 5 \cdot n$. Assume that $n = p_1 p_2 \cdots p_s$, where $s \geq 3$ and p_i 's are distinct primes.

We first consider the case where A is soluble.

Lemma 4.1. *Assume that A is soluble. Then no graph Γ exists.*

Proof. Let F be the Fitting subgroup of A . By Lemma 2.1, $F \neq 1$ and $C_A(F) \leq F$. Further, $F = O_2(A) \times O_{p_1}(A) \times O_{p_2}(A) \times \cdots \times O_{p_s}(A)$, where $O_2(A)$, $O_{p_1}(A)$, $O_{p_2}(A), \dots, O_{p_s}(A)$ denote the largest normal 2-, p_1 -, p_2 -, \dots , p_s -subgroups of A , respectively.

For each $p_i \in \{p_1, p_2, \dots, p_s\}$, $O_{p_i}(A)$ has at least three orbits on $V\Gamma$, by Theorem 2.6, $O_{p_i}(A)$ is semi-regular on $V\Gamma$. Therefore, F is semi-regular on $V\Gamma$ and $O_{p_i}(A) \leq \mathbb{Z}_{p_i}$. This argument also proves $O_2(A) \leq \mathbb{Z}_4$ or \mathbb{Z}_2^2 . If $O_2(A) = \mathbb{Z}_4$ or \mathbb{Z}_2^2 , then by Theorem 2.6, the normal quotient graph $\Gamma_{O_2(A)}$ is a pentavalent symmetric graph of odd order, which is a contradiction. Thus, $O_2(A) \leq \mathbb{Z}_2$, $F \cong \mathbb{Z}_m$, where $m \mid 2n$. It implies that $C_A(F) \geq F$, and so $C_A(F) = F$.

If F has at least three orbits on $V\Gamma$, then, by Theorem 2.6, Γ_F is A/F -arc-transitive. Since $A/F = A/C_A(F) \leq \text{Aut}(F)$ is abelian, we have $(A/F)_\delta = 1$, where $\delta \in V\Gamma_F$, which is a contradiction.

Thus, F has at most two orbits on $V\Gamma$. If F is transitive on $V\Gamma$, then F is regular on $V\Gamma$. It follows that $\Gamma \cong \text{Cay}(F, S)$ is a normal arc-transitive Cayley graph of F . Then we easily conclude that S consists of the involutions of F . Since F has at most one involution, it follows that $F = \langle S \rangle \leq \mathbb{Z}_2$, which is a contradiction.

Hence F has two orbits on $V\Gamma$ and $F \cong \mathbb{Z}_{2n}$ and $K := O_{p_3}(A) \times O_{p_4}(A) \times \cdots \times O_{p_s}(A) \cong \mathbb{Z}_{p_3 p_4 \cdots p_s}$. Since $K \trianglelefteq A$ has $4p_1 p_2$ orbits on $V\Gamma$, by Theorem 2.6(i), Γ_K is an A/K -arc-transitive pentavalent graph of order $4p_1 p_2$, and hence Γ_K satisfies Lemma 2.7. Since A/K is soluble, by Remark 2.8, a contradiction occurs. \square

We next consider the case where A is insoluble.

Lemma 4.2. *Assume that A is insoluble and has no nontrivial soluble normal subgroup. Then $\text{Aut}\Gamma \cong J_1$, $\text{PSL}(2, p)$ or $\text{PGL}(2, p)$. Further, If $\text{Aut}\Gamma \cong J_1$, then $\Gamma \cong \mathcal{C}_{17556}^i$ which satisfy the conditions in Row 2 to Row 5 of Table 1 of Theorem 1.1, where $1 \leq i \leq 5$.*

Proof. Let N be a minimal normal subgroup of A . Then $N = S^d$, where S is a nonabelian simple group and $d \geq 1$.

If N has more than three orbits on $V\Gamma$, then by Theorem 2.6, N is semi-regular on $V\Gamma$ and so $|N|$ divides $4n$. Since N is a direct product of nonabelian groups, it implies that 4 divides $|N|$. Again by Theorem 2.6, Γ_N is a pentavalent symmetric graph of odd order n , a contradiction. Hence, N has at most two orbits on $V\Gamma$, so $2n$ divides $|N|$.

Moreover, since $p_s > 5$, p_s divides $|N|$ and p_s^2 does not divide $|N|$ as $|A| \mid 2^{11} \cdot 3^2 \cdot 5 \cdot p_1 p_2 \cdots p_s$, we conclude that $d = 1$ and $N = S$ is a nonabelian simple group. Let $C = \mathbf{C}_A(S)$. Then $C \triangleleft A$, $C \cap S = 1$ and $\langle C, S \rangle = C \times S$. If $C \neq 1$, then C is insoluble as $C \triangleleft A$ and A has no soluble normal subgroup. It follows that 4 divides $|C|$. A similarly argument with the above paragraph, we have $2n$ divides $|C|$. Hence $4n^2$ divides $|A| = 2^{11} \cdot 3^2 \cdot 5 \cdot n$, and so n divides $2^9 \cdot 3^2 \cdot 5$. It implies that $n = 3 \cdot 5$, a contradiction with n having at least three prime factors. So $C = 1$ and $A = A/C \leq \text{Aut}(S)$, that is, A is almost simple with socle S .

If $S_\alpha = 1$, then S acts regularly on $V\Gamma$. Hence S is a non-abelian simple group such that $|S| = 4n$. By checking the orders of nonabelian simple groups (see [6, P. 135-136] for example), we have that $S = \text{PSL}(2, p)$ and so $A \leq \text{Aut}(S) = \text{PGL}(2, p)$, which is impossible as A is transitive on $A\Gamma$, $|A| \leq 2|S|$ and $|A\Gamma| = 5|S|$. Hence $S_\alpha \neq 1$. Since Γ is connected and $S \triangleleft A$, we have $1 \neq S_\alpha^{\Gamma(\alpha)} \triangleleft A_\alpha^{\Gamma(\alpha)}$, it follows that $5 \mid |S_\alpha|$, we thus have $10 \cdot p_1 p_2 \cdots p_s$ divides $|S|$.

Thus, $\text{soc}(A) = S$ is a nonabelian simple group such that $|S| \mid 2^{11} \cdot 3^2 \cdot 5 \cdot n$ and $10 \cdot n \mid |S|$. Hence the triple $(S, |S|, n)$ lies in Table 2 of Lemma 2.5. We will analyse all the candidates one by one in the following.

If $\cong \text{PSL}(2, p)$ with $p \geq 29$ a prime, then $A \cong \text{PSL}(2, p)$ or $\text{PGL}(2, p)$, the Lemma holds. If $(S, n) = (J_1, 3 \cdot 7 \cdot 11 \cdot 19)$, then $|V\Gamma| = 17556$ and $A \cong J_1$ as $\text{Out}(J_1) = 1$. It then follows from Example 3.1 that $\Gamma \cong \mathcal{C}_{17556}^i$ which satisfy the conditions in Row 1 to Row 5 of Table 1 of Theorem 1.1, where $1 \leq i \leq 5$.

Assume $(S, n) = (\text{Sz}(32), 5 \cdot 31 \cdot 41)$. Since $\text{Out}(\text{Sz}(32)) \cong \mathbb{Z}_5$ (see Atlas [3] for example), $A \cong \text{Sz}(32)$ or $\text{Sz}(32) \cdot \mathbb{Z}_5$, so $|A_\alpha| = \frac{|A|}{4n} = 1280$ or 6400 , which is not possible by Lemma 2.4. Similarly, for the case $(S, n) = (\text{PSL}(5, 2), 3 \cdot 7 \cdot 31)$, then $A \cong \text{PSL}(5, 2)$ or $\text{PSL}(5, 2) \cdot \mathbb{Z}_2$ as $\text{Out}(\text{PSL}(5, 2)) \cong \mathbb{Z}_2$. Thus, $|A_\alpha| = \frac{|A|}{4n} = 3840$ or 7680 , which is impossible by Lemma 2.4. For the case where $(S, n) = (\text{PSL}(2, 2^8), 3 \cdot 17 \cdot 257)$, since $A \cong \text{PSL}(2, 2^8) \cdot O$, where $O \leq \text{Out}(\text{PSL}(2, 2^8)) \cong \mathbb{Z}_8$, we have $|A_\alpha| = \frac{|A|}{4n} = 2^k \cdot 5$, where $6 \leq k \leq 9$, which is also impossible by Lemma 2.4. For the case where $(S, n) = (\text{PSU}(3, 4), 3 \cdot 5 \cdot 13)$, since $A \cong \text{PSU}(3, 4) \cdot O$, where $O \leq \text{Out}(\text{PSU}(3, 4)) \cong \mathbb{Z}_4$, we have $|A_\alpha| = \frac{|A|}{4n} = 2^k \cdot 5$, where $4 \leq k \leq 6$, which is impossible by Lemma 2.4.

Assume $(S, n) = (\text{PSp}(4, 4), 3 \cdot 5 \cdot 17)$. Since $S \leq A \leq \text{Aut}(S) \cong \text{PSp}(4, 4) \cdot \mathbb{Z}_4$, we have $|A_\alpha| = \frac{|A|}{4n} = 960$, 1920 or 3840 . If $|A_\alpha| = 960$ or 1920 , then by Lemma 2.4, $A_\alpha \cong \text{ASL}(2, 4)$ or $\text{A}\Sigma\text{L}(2, 4)$. However, by Atlas [3], $\text{PSp}(4, 4)$ has no subgroup

isomorphic to $\text{ASL}(2, 4)$ and $\text{PSp}(4, 4) \cdot \mathbb{Z}_2$ has no subgroup isomorphic to $\text{A}\Sigma\text{L}(2, 4)$. If $|\mathbf{A}_\alpha| = 3840$, then also by Lemma 2.4, a contradiction occurs.

Assume $(S, n) = (\text{PSL}(2, 2^6), 3 \cdot 7 \cdot 13)$. Recall that S has at most two orbits on $V\Gamma$, $|S_\alpha| = \frac{|S|}{4n} = 240$ or $\frac{|S|}{2n} = 480$. However, by Lemma 2.2, $\text{PSL}(2, 2^6)$ has no maximal subgroup with order a multiple of 240, a contradiction occurs. Similarly, for the case $(S, n) = (\text{J}_2, 3 \cdot 5 \cdot 7)$. Then $|S_\alpha| = \frac{|S|}{4n} = 2880$ or $\frac{|S|}{2n} = 5760$. By Atlas [3], J_2 has no maximal subgroup with order a multiple of 2880, a contradiction also occurs.

Assume $S \cong \text{M}_{23}$. Then $n = 3 \cdot 7 \cdot 11 \cdot 23$ or $7 \cdot 11 \cdot 23$, and as $\text{Out}(\text{M}_{23}) = 1$, we have $\mathbf{A} = S$ and $|\mathbf{A}_\alpha| = \frac{|\text{M}_{23}|}{4n} = 480$ or 1440 . By Lemma 2.4, it is impossible for the case $|\mathbf{A}_\alpha| = 480$. For the latter case, by a direct computation using Magma [1], no graph Γ exists. If $(S, n) = (\text{M}_{22}, 7 \cdot 11 \cdot 23)$, as $\text{Out}(\text{M}_{22}) \cong \mathbb{Z}_2$, we have $\mathbf{A} \cong \text{M}_{22}$ or $\text{M}_{22} \cdot \mathbb{Z}_2$, so $|\mathbf{A}_\alpha| = \frac{|\mathbf{A}|}{4n} = 480$ or 960 , a computation by Magma [1] shows that no graph Γ exists. Similarly, we can exclude the case where $(S, n) = (\text{PSL}(2, 25), 3 \cdot 5 \cdot 13)$ by Magma [1].

Finally, assume $(S, n) = (\text{M}_{24}, 3 \cdot 7 \cdot 11 \cdot 23)$ or $(\text{J}_1, 3 \cdot 7 \cdot 11 \cdot 19)$. Since $\text{Out}(\text{M}_{24}) = \text{Out}(\text{J}_1) = 1$, we always have $\mathbf{A} = S$. Hence $|\mathbf{A}_\alpha| = \frac{|\mathbf{A}|}{4n} = 11520$ or 10 . A computation by Magma [1] also shows that no graph Γ exists. \square

We next assume that \mathbf{A} has a nontrivial soluble normal subgroup. Let N be a minimal soluble normal subgroup of \mathbf{A} . Then there exists a prime $r \mid 4n$ such that $N \cong \mathbb{Z}_r^d$. Further, N has at least three orbits on $V\Gamma$. It follows from Theorem 2.6 that N is semi-regular on $V\Gamma$, and so $|N| = |\mathbb{Z}_r|^d \mid |V\Gamma| = 4n$. If $d \geq 2$, then $(r, d) = (2, 2)$. It follows that Γ_N is an arc-transitive graph of odd order, a contradiction. Hence $d = 1$, $N = \mathbb{Z}_r$. The next lemma consider the case where $r = 2$.

Lemma 4.3. *Assume that \mathbf{A} is insoluble and has a minimal soluble normal subgroup $N = \mathbb{Z}_2$. Then one of the following statements holds:*

- (1) $\text{Aut}\Gamma \cong \text{PSL}(2, p) \times \mathbb{Z}_2$ or $\text{PGL}(2, p) \times \mathbb{Z}_2$, where $p \geq 29$ is a prime.
- (2) $\text{Aut}\Gamma \cong \text{PSL}(2, 25) \times \mathbb{Z}_2$ and Γ is isomorphic to \mathcal{C}_{780}^i in Table 1, where $1 \leq i \leq 2$.
- (3) $\text{Aut}\Gamma \cong \text{J}_1 \times \mathbb{Z}_2$ and Γ is isomorphic to \mathcal{C}_{5852} in Table 1.

Proof. Since N has more than three orbits on $V\Gamma$, then by Theorem 2.6, Γ_N is an \mathbf{A}/N -arc-transitive pentavalent graph of order $\bar{n} = 2n$. It follows that Γ_N is isomorphic to one of the graphs in Lemma 2.8. Since $\mathbf{A}/N \leq \text{Aut}\Gamma_N$ and \mathbf{A}/N is insoluble, we have that $\text{Aut}\Gamma_N$ is insoluble and so $\text{Aut}\Gamma_N \cong \text{PSL}(2, p)$, $\text{PGL}(2, p)$, $\text{PSL}(2, 25)$ or J_1 . Let $\bar{\mathbf{A}} := \text{Aut}\bar{\Gamma}$.

Suppose that $\bar{\mathbf{A}} \cong \text{PSL}(2, p)$ or $\text{PGL}(2, p)$. Since \mathbf{A}/N is insoluble, by Lemma 2.2 and Lemma 2.3, \mathbf{A}/N is isomorphic to A_5 , $\text{PSL}(2, p)$ or $\text{PGL}(2, p)$. Since \mathbf{A}/N is transitive on $\mathbf{A}\Gamma_N$, we can further conclude that \mathbf{A}/N is isomorphic to $\text{PSL}(2, p)$ or $\text{PGL}(2, p)$. Therefore, $\mathbf{A} \cong N \cdot \text{PSL}(2, p)$ or $N \cdot \text{PGL}(2, p)$, that is, $\mathbf{A} \cong \text{PSL}(2, p) \times \mathbb{Z}_2$, $\text{SL}(2, p)$, $\text{PGL}(2, p) \times \mathbb{Z}_2$ or $\text{SL}(2, p) \cdot \mathbb{Z}_2$. Assume first that $\mathbf{A} \cong \text{SL}(2, p)$. Note that $\text{SL}(2, p)$ has a unique central involution. Then by Lemma 2.4, $\mathbf{A}_\alpha \cong \mathbb{Z}_5$. It follows that $|V\Gamma| = |\mathbf{A} : \mathbf{A}_\alpha|$ is divisible by 8 as $|\text{SL}(2, p)|$ is divisible by 8, a contradiction. Assume next that $\mathbf{A} \cong \text{SL}(2, p) \cdot \mathbb{Z}_2$. Then \mathbf{A} contains a normal subgroup H

isomorphic to $\text{SL}(2, p)$. Since $8 \mid |H|$, we have $H_\alpha \neq 1$. By Theorem 2.6, H has at most two orbits on $V\Gamma$ and so $\frac{|A_\alpha|}{|H_\alpha|} \mid 2$. If H is transitive on $V\Gamma$, then H is arc-transitive. A similar argument with the case $A \cong \text{SL}(2, p)$, a contradiction occurs. Therefore, H has two orbits on $V\Gamma$ and so $H_\alpha = A_\alpha$. Since H has a unique central involution, by Lemma 2.4, $A_\alpha \cong \mathbb{Z}_5$, it follows that $|V\Gamma| = |A : A_\alpha|$ is divisible by 16, a contradiction. Therefore, $A \cong \text{PSL}(2, p) \times \mathbb{Z}_2$ or $\text{PGL}(2, p) \times \mathbb{Z}_2$ in this case.

Suppose that $\bar{A} \cong \text{PSL}(2, 25)$. Since Γ_N is A/N -arc-transitive, we have that $5 \cdot 390 \mid |A/N|$. By checking the maximal subgroup of $\text{PSL}(2, 25)$ (see Atlas [3] for example), we have that $A/N = \bar{A} \cong \text{PSL}(2, 25)$. It follows that $A \cong \text{SL}(2, 25)$ or $\text{PSL}(2, 25) \times \mathbb{Z}_2$. If $A \cong \text{PSL}(2, 25) \times \mathbb{Z}_2$, then by Example 3.1, $\Gamma \cong \mathcal{C}_{780}^i$ in Table 1, where $1 \leq i \leq 2$. If $A \cong \text{SL}(2, 25)$, then by Magma [1], no graph exists.

Suppose that $\bar{A} \cong J_1$. Similarly, since Γ_N is A/N -arc-transitive, we have that $5 \cdot 2926 \mid |A/N|$. By checking the maximal subgroup of J_1 (see Atlas [3] for example), we have that $A/N = \bar{A} \cong J_1$. Since the Schur multiplier of J_1 is \mathbb{Z}_1 , $A \cong N.J_1 \cong J_1 \times \mathbb{Z}_2$. By Example 3.1, $\Gamma \cong \mathcal{C}_{5852}$ in Table 1. \square

Finally, suppose that $r > 2$. We first prove the following lemma.

Lemma 4.4. *Assume that Σ is isomorphic to one of the graphs listed in Lemma 2.7, in Lemma 4.2 and in Lemma 4.3. If M is an arc-transitive subgroup of $\text{Aut}\Sigma$, then M contains a subgroup isomorphic to $\text{PSL}(2, p)$, J_1 , $\text{PSL}(2, 25)$ or A_5 .*

Proof. By checking the graphs in Lemma 2.7, in Lemma 4.2 and in Lemma 4.3, we have that $\text{Aut}\Sigma$ is isomorphic to one of the groups $\text{PSL}(2, p)$, $\text{PGL}(2, p)$, $\text{PSL}(2, p) \times \mathbb{Z}_2$, $\text{PGL}(2, p) \times \mathbb{Z}_2$, J_1 , $J_1 \times \mathbb{Z}_2$, $\text{PSL}(2, 25) \times \mathbb{Z}_2$ or $A_5 \times D_{10}$. If $\text{Aut}\Sigma \cong \text{PSL}(2, p)$, then since $p \mid n$ and $20n \mid |M|$, by Lemma 2.2, $M \leq \mathbb{Z}_p : \mathbb{Z}_{\frac{p-1}{2}}$ or $M = \text{Aut}\Sigma \cong \text{PSL}(2, p)$. If $M \leq \mathbb{Z}_p : \mathbb{Z}_{\frac{p-1}{2}}$, then $M \cong \mathbb{Z}_p : \mathbb{Z}_l$ for some $l \mid \frac{p-1}{2}$. Thus, M has a normal subgroup, say $S \cong \mathbb{Z}_p$, which has more than three orbits on $V\Sigma$. It then follows from Theorem 2.6 that the normal quotient graph Σ_S is M/S -arc-transitive, a contradiction occurs as $M/S \cong \mathbb{Z}_l$ is cyclic. Hence, $M \not\leq \mathbb{Z}_p : \mathbb{Z}_{\frac{p-1}{2}}$ and so $M = \text{Aut}\Sigma \cong \text{PSL}(2, p)$. If $\text{Aut}\Sigma \cong \text{PGL}(2, p)$, then since $20n \mid |M|$, by Lemma 2.3, $M \leq \mathbb{Z}_p : \mathbb{Z}_{p-1}$, $M \leq \text{PSL}(2, p)$ or $M = \text{Aut}\Sigma \cong \text{PGL}(2, p)$. A similar argument, we can conclude that $M \geq \text{PSL}(2, p)$. Similarly, we can further show that $M \geq \text{PSL}(2, p)$ for the case $\text{Aut}\Sigma \cong \text{PSL}(2, p) \times \mathbb{Z}_2$ or $\text{PGL}(2, p) \times \mathbb{Z}_2$.

If $\text{Aut}\Sigma \cong J_1$, then since Σ is isomorphic to \mathcal{C}_{2926} , we have that $5 \cdot 2926 \mid |M|$. By checking the maximal subgroup of J_1 (see Atlas [3]), we have that $M = \text{Aut}\Sigma \cong J_1$. We can further show that $M \geq J_1$ for the case $\text{Aut}\Sigma \cong J_1 \times \mathbb{Z}_2$, $M \geq \text{PSL}(2, 25)$ for the case $\text{Aut}\Sigma \cong \text{PSL}(2, 25) \times \mathbb{Z}_2$ and $M \geq A_5$ for the case $\text{Aut}\Sigma \cong A_5 \times D_{10}$. \square

Now assume that A has a minimal soluble normal subgroup $N = \mathbb{Z}_r$ for $r > 2$.

Lemma 4.5. *Assume that A has a minimal soluble normal subgroup $N = \mathbb{Z}_r$ for $r > 2$. Then the normal quotient Γ_N is not isomorphic to any of the graphs Σ listed in Lemma 4.4.*

Proof. Suppose to the contrary that Γ_N is isomorphic to one of the graphs in Lemma 4.4. Then by Theorem 2.6, $A/N \leq \text{Aut}\Gamma_N$ is transitive on $A\Gamma_N$. Let $\Omega := \{\text{PSL}(2, p), J_1, \text{PSL}(2, 25), A_5\}$. It follows from Lemma 4.4 that there exists a

subgroup M/N of A/N isomorphic to one of the groups in Ω . Since now $M/N \leq A/N \leq \text{Aut}\Gamma_N$, it follows from the structure of $\text{Aut}\Gamma_N$ that $M/N \trianglelefteq A/N$. Therefore, $M' \text{char} M \trianglelefteq A$, it implies that $M' \trianglelefteq A$. On the other hand, since the order of the Schur multiplier of a group in Ω is less than or equal to 2 (see [10, Theorem 7.1.1] for $\text{PSL}(2, p)$ and Atlas [3] for the others) and $r > 2$, we have that $M' \in \Omega$ and $4 \mid |M'|$. If M' has more than three orbits on $V\Gamma$, then by Theorem 2.6, $\Gamma_{M'}$ is a pentavalent symmetric graph of odd order, a contradiction. Thus, M' has at most two orbits on $V\Gamma$ and so $2n$ divides $|M'|$. Let $\bar{A} := \text{Aut}\Gamma_N$, $\bar{n} := \frac{n}{r}$ and $\bar{M} := M/N$. Then $M' \cong \bar{M}$.

Assume first that $r > 5$. Then since $|\bar{M}| \mid |\bar{A}| \mid 2^{11} \cdot 3^2 \cdot 5 \cdot \frac{n}{r}$ and n is an odd square-free integer, we have that r does not divide $|\bar{M}| = |M'|$. It implies that M' has at least r orbits on $V\Gamma$, a contradiction.

Assume next that $r = 3$. Since $\bar{M} \cong M'$ has at most two orbits on $V\Gamma_N$ (if not $(\Gamma_N)_{\bar{M}}$ is a pentavalent symmetric graph of odd order, a contradiction), we have that $|\bar{M} : \bar{M}_\delta| = 2\bar{n}$ or $4\bar{n}$, where $\delta \in V\bar{\Gamma}$. Now $2n$ divides $|\bar{M}|$ and $|\bar{M} : \bar{M}_\delta| = \frac{2n}{r}$ or $\frac{4n}{r}$. It implies that $r = 3$ divides \bar{M}_δ . Therefore $3 \mid |\bar{A}_\delta|$. By Lemma 2.4, \bar{A}_δ is nonsolvable, because $|\bar{A}_\delta|$ does not divide 80, forcing that \bar{M}_δ is nonsolvable. If $\bar{M} \cong \text{PSL}(2, p)$, then by Lemma 2.2, $\bar{M}_\delta \cong A_5$. Hence $M'_\alpha \leq (M'N)_\alpha \cong (M'N/N)_\delta = \bar{M}_\delta \cong A_5$ by Theorem 2.6 (ii). Note that $|M'_\alpha| = 20$, it contradicts that A_5 has no subgroup of order 20. If $\bar{M} \cong J_1$, then $\Gamma_N \cong C_{5852}$ or C_{17556}^i in Table 1, where $1 \leq i \leq 5$. If $\Gamma_N \cong C_{17556}^i$, then $\bar{A}_\delta \cong D_{10}$ is soluble, a contradiction. If $\Gamma_N \cong C_{5852}$, then $\bar{M}_\delta = \bar{A}_\delta \cong A_5$. A similar argument with the case $\bar{M} \cong \text{PSL}(2, p)$ leads to a contradiction. If $\bar{M} \cong A_5$, then $\Gamma_N \cong C_{60}$ in Table 3 and $\bar{A}_\delta \cong D_{10}$ is soluble, a contradiction. If $\bar{M} \cong \text{PSL}(2, 25)$, then $\Gamma_N \cong C_{780}^1$ or C_{780}^2 in Table 1 and $\bar{A}_\delta \cong F_{20}$ is soluble, also a contradiction.

Finally assume that $r = 5$. Then $|M' : M'_\alpha| = 2n$ or $4n$ as M' has at most two orbits on $V\Gamma$. Since Γ is connected and $1 \neq M'_\alpha \triangleleft A_\alpha$, we have $1 \neq M'_\alpha \triangleleft^{\Gamma(\alpha)} A_\alpha^{\Gamma(\alpha)}$, it follows that $5 \mid |M'_\alpha|$. On the other hand, since \bar{M} has at most two orbits on $V\Gamma_N$, we have that $|\bar{M} : \bar{M}_\delta| = 2\bar{n}$ or $4\bar{n}$. Note that $\bar{M} \cong M'$ and $r = 5$. Hence $\frac{|\bar{M}_\delta|}{|M'_\alpha|} = 5$, it follows that $5^2 \mid |\bar{M}_\delta| \mid |\bar{A}_\delta|$, a contradiction with $|\bar{A}_\delta| \mid 2^9 \cdot 3^2 \cdot 5$ by Lemma 2.4, a contradiction. \square

The final lemma completes the proof of Theorem 1.1.

Lemma 4.6. *Assume A is insoluble. Then A has no minimal soluble normal subgroup isomorphic to \mathbb{Z}_r with $r > 2$.*

Proof. Suppose that, on the contrary, A has a minimal soluble normal subgroup $N = \mathbb{Z}_r$ with $r > 2$. We prove the lemma by induction on the order of Γ .

Assume first that $n = pqt$ has three prime factors (Note that, by Table 3, the conclusion of Lemma 4.6 does not hold for $n = pq$). Without loss of generality, we may assume that $r = t$. Then Γ_N is a pentavalent symmetric graph of order $4pq$. By Lemma 2.7, Γ_N is isomorphic to one of the graphs in Table 3, which contradicts to Lemma 4.5.

Assume next that n has at least four prime factors. Note that $\text{Aut}\Gamma_N$ is insoluble. If $\text{Aut}\Gamma_N$ has no nontrivial soluble normal subgroup, then Γ_N is isomorphic to one of the graphs in Lemma 4.2, which contradicts to Lemma 4.5. If $\text{Aut}\Gamma_N$ has a

minimal soluble normal subgroup \bar{N} , then we can also conclude that $\bar{N} \cong \mathbb{Z}_f$ with f a prime. If $f > 2$, then by induction, no such Γ_N exists, a contradiction. If $f = 2$, then Γ_N is isomorphic to one of the graphs in Lemma 4.3, which also contradicts to Lemma 4.5. This completes the proof of the lemma. \square

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